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A comparison between analytical calculations of the shakedown load by the bipotential approach and step-by-step computations for elastoplastic materials with nonlinear kinematic hardening

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Abstract

The class of generalized standard materials is not relevant to model the nonassociative constitutive equations. The bipotential approach, based on a possible generalization of Fenchel's inequality, allows the recovery of the flow rule normality in a weak form of an implicit relation. This defines the class of implicit standard materials. For such behaviours, this leads to a weak extension of the classical bound theorems of the shakedown analysis. In the present paper, we recall the relevant features of this theory. Considering an elastoplastic material with nonlinear kinematic hardening rule, we apply it to the problem of a sample in plane strain conditions under constant traction and alternating torsion in order to determine analytically the interaction curve bounding the shakedown domain. The aim of the paper is to prove the exactness of the solution for this example by comparing it to step-by-step computations of the elastoplastic response of the body under repeated cyclic loads of increasing level. A reliable criterion to stop the computations is proposed. The analytical and numerical solutions are compared and found to be closed one of each other. Moreover, the method allows uncovering an additional '2 cycle shakedown curve' that could be useful for the shakedown design of structure.

Keywords: Nonassociative plasticity; Nonlinear kinematic hardening; Implicit standard materials; Shakedown; Bipotential

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1. Introduction

Many engineering structures or structural elements are submitted to cyclic mechanical loads and/or temperature fields, acting simultaneously.

For this kind of loading, the limit load, based on proportional loads, do not ensure the structural safety. The experimental tests show that beyond the limit load, structures subjected to cyclic loading may fail after a finite number of cycles by accumulation of plastic strains, called ratchet or incremental collapse, or by periodic elastoplastic strain response, called plastic shakedown or accommodation. Nevertheless, the structure can also endure a very large number of cycles (high-cycle fatigue). This behaviour, called elastic shakedown, corresponds to a stabilization of the plastic strains. In this case, the response of the structure becomes purely elastic.

In the pioneering works on shakedown theory by Melan (1936, 1938) and Koiter (1960) the plastic effects covered by the theoretical development were restricted to linear elastic perfectly plastic materials, with no thermal influence and in quasi-static processes. From the engineering point of view, these assumptions were not always realistic. Consequently, extensions of the classical shakedown theorems have attracted much interest in the last years. In particular, the influences of hardening, geometrical effects and damage, thermal loadings, dynamic loadings were the aim of many publications (see, for example, the review of Maier et al., 2000 and more recently, Nguyen, 2003; Weichert and Hachemi, 1998; Borino, 2000; Pham, 2003).

Now, we discuss how the choice of the constitutive law has a crucial influence on the kind of collapse which can be represented (Fig. 1). Although simple, the standard elastic perfectly plastic model allows representing both incremental and alternating plasticity collapses. The modern trends to use harden metals exhibiting higher failure strength needs to take into account the hardening effects. From the viewpoint of the shakedown analysis, it requires a careful choice of the suitable model. Prager's kinematic hardening is the simplest one because of its linearity. Unfortunately, under variable repeated loading, it does not allow to represent the incremental collapse (see for example, Dang Van et al., 2002). Thus, its use is restricted to particular engineering applications for which ones the Codes only require to verify the structure safety against alternating plasticity collapse. On the other hand, the limited linear kinematic hardening model using a two-surface yield condition is able to represent the incremental collapse too and provides a powerful tool for shakedown analysis. This model was extensively used by Weichert and his collaborators for numerical applications in shakedown theory (see Weichert and Hachemi, 1998; Dang Van et al., 2002). Nevertheless, the previous hardening being linear under the limit surface, leads to a rather rough idealization of the plasticity limit cycles observed in experimental testing.

According to many experimenters, a more realistic representation of the cyclic plasticity of metals is given by the so-called nonlinear kinematic hardening rules. A simple and efficient one was proposed by Armstrong and Frederick (1966) and was popularized among others by Lemaitre and Chaboche (1990) (see also Chaboche, 1991). The main drawback of this kind of model is the nonassociated nature of the constitutive law.

The earlier extension of the shakedown criteria to nonassociative rules (Maier, 1969) concerned perfect plasticity with piecewise-linear yield functions. It was built on the concept of reduced elastic domain and the description by a plastic potential distinct from the yield function. This extension was again recently examined by Pycko and Maier (1995) and Corigliano et al. (1995) for elastic–plastic materials.

In a more general context, another interesting viewpoint is provided by the elastic sanctuary concept proposed by Nayroles and Weichert (1993).

On the other hand, an alternative approach to nonassociated plasticity stems from the implicit standard material model. By analogy with the standard material model (Halphen and Nguyen, 1975), this model was introduced to recovery the flow rule normality, in a weak form of an implicit relation (in the sense of the implicit function theorem) (De Saxcé, 1992; De Saxcé and Bousshine, 1998; Dang Van et al., 2002). One of the original results of the implicit standard material theory is to prove that many nonstandard dissipative

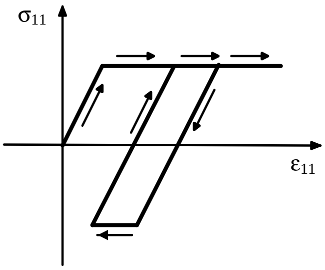
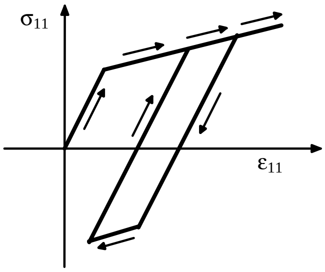
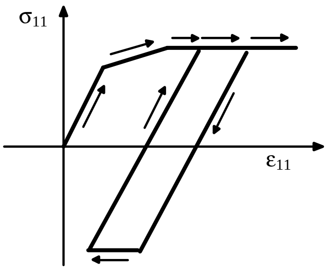
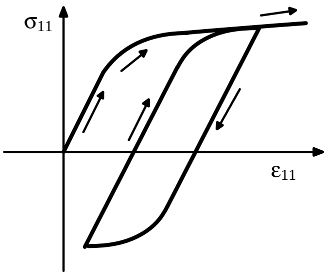
loading curve	model	collapse	
		plastic shakedown (accommodation)	incremental collapse (ratchet)
	elastic perfectly plastic model	yes	yes
	Prager's linear kinematic hardening	yes	no
	limited linear kinematic hardening	yes	yes
	non linear kinematic hardening	yes	yes

Fig. 1. Constitutive law and represented collapse.

laws can be simply represented by a suitable pseudo-potential depending on the dual variables, internal variable rates and associated variables. The properties of the so-called bipotential are based on an extension of Fenchel's inequality (Fenchel, 1949) and allow, in the framework of convex analysis, the generalization of Moreau's superpotential (Moreau, 1968).

Within the frame of bipotential theory, we proposed in (De Saxcé et al., 2000) a generalization of Koiter's bound theorems for implicit standard materials. As application, the problem of a sample subjected to constant tension and alternating torsion in plane stress state was considered. The shakedown load, denoted λ^{a} in this paper, calculated by the bipotential approach is exactly the same as the one proposed by Lemaitre and Chaboche (1990). A similar formula was obtained in Bodovillé and de Saxcé (2001) for an improved version of the nonlinear kinematic hardening rule with threshold. The two previous works seems promising but it worthwhile pointing out that the exactness of the result is not absolutely ensured. The crux of the mater is that the extended bound theorems are stated for the new class of nonassociated materials but under the condition that time independent residual stress fields exist and characterize the shakedown solution. This was rigorously proved by Melan (1936) for the standard elastic perfectly materials. Up to now, a similar theorem concerning the residual stress fields cannot be stated and proved for the class of implicit standard materials.

The aim of the paper is to prove the exactness of the solution at least for one example by comparing it to separate step-by-step computations of the elastoplastic response of the body under repeated cyclic loads of increasing level. In other word, we perform "numerical testing" in order to approach the shakedown load. The main difficulty is to determine a reliable criterion to stop the computations, taking into account that the asymptotic solution cannot be reached in a finite number of cycle. Experimental investigations on shakedown were carried out by various authors. Without being exhaustive, we can quote Massonnet (1956) and Massonnet and Save (1976) for continuous mild-steel beams, Proctor and Flinders (1967) for spherical pressure vessels with radial and oblique nozzles, Armstrong and Townley (1967) for the bending of T beams, Findlay et al. (1971) for torispherical pressure vessel heads, Townley et al. (1970–1971) for torispherical pressure vessels, Ceradini et al. (1975) for steel orthotropic plates, Leers et al. (1985) for tubes under thermo-mechanical loads. The last work undoubtedly presents the most sophisticated methodology for detecting the shakedown load by observing that the displacement increment falls down and then becomes higher again. We did not find again this typical behaviour. Then, we develop our own shakedown detection method as described latter on.

The organization of the paper is as follows. In Section 2, we recall some theoretical basis of the bipotential approach and its application to shakedown analysis. In Section 3, we consider the problem of a sample under constant tension and alternating torsion, extending to the plane strain the results in De Saxcé et al. (2000) and Lemaitre and Chaboche (1990). We find an analytical formula defining the shakedown interaction curve in the traction stress–maximum shear stress space. In Section 4, we present an alternative procedure to determine the shakedown curve. It is based on numerical step-by-step computations of the sample response under repeated cyclic loads and an efficient criterion to stop the computations. In Section 5, the analytical and numerical results are compared and found to be closed one of each other. Incidentally, our method allows uncovering -that was surprising for us—an additional interaction curve (called '2 cycle shakedown curve' in the paper) that could be useful for the shakedown design of structure.

2. Bipotential approach for shakedown analysis

2.1. The bipotential concept

Using bipotential, it has been shown (De Saxcé et al., 2000; Dang Van et al., 2002) that many nonstandard dissipative materials are in fact governed by a normality rule, but in an implicit sense.

Let the generalized velocities be $\dot{\boldsymbol{\kappa}} = (\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\kappa}}') \in V$, the velocity space, including the velocities $\dot{\boldsymbol{\kappa}}'$ of additional internal variables (hardening, damage...), and the corresponding associated variables $\boldsymbol{\pi} = (\boldsymbol{\sigma}, \boldsymbol{\pi}') \in F$, the stress space. The spaces V and F are equipped with locally convex topologies compatible with the duality expressed by a bilinear form $(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}) \mapsto \dot{\boldsymbol{\kappa}} \cdot \boldsymbol{\pi}$.

A bipotential is a function b from $V \times F$ into $]-\infty, \infty]$, separately convex, satisfying the fundamental inequality generalizing Legendre–Fenchel one (Fenchel, 1949):

$$\forall(\dot{\boldsymbol{\kappa}}^*, \boldsymbol{\pi}^*) \in V \times F, \quad b(\dot{\boldsymbol{\kappa}}^*, \boldsymbol{\pi}^*) \geqslant \dot{\boldsymbol{\kappa}}^* \cdot \boldsymbol{\pi}^* \tag{1}$$

The couples $(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi})$, for which the variables are related by the dissipative law, are qualified as *extremal* in the sense that the equality is reached in the previous relation:

$$b(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}) = \dot{\boldsymbol{\kappa}} \cdot \boldsymbol{\pi} \tag{2}$$

From (1) and (2), we deduce the following inequalities to be satisfied by the extremal couples:

$$\forall \boldsymbol{\pi}^* \in F, \quad b(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}^*) - b(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}) \geqslant \dot{\boldsymbol{\kappa}} \cdot (\boldsymbol{\pi}^* - \boldsymbol{\pi})$$
$$\forall \dot{\boldsymbol{\kappa}}^* \in V, \quad b(\dot{\boldsymbol{\kappa}}^*, \boldsymbol{\pi}) - b(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}) \geqslant (\dot{\boldsymbol{\kappa}}^* - \dot{\boldsymbol{\kappa}}) \cdot \boldsymbol{\pi}$$

Briefly, they are characterized by the following differential inclusions:

$$\dot{\boldsymbol{\kappa}} \in \partial_{\pi} b(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}), \quad \boldsymbol{\pi} \in \partial_{\dot{\kappa}} b(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi})$$

where ∂_{π} ($\partial_{\dot{\kappa}}$ respectively) denotes the subdifferential when partial derivating with respect to $\boldsymbol{\pi}$ (respectively $\dot{\boldsymbol{\kappa}}$). For elastic–plastic behaviour laws, the set of extremal couples is equivalent to that of the material states satisfying the plastic flow rule. Physically, the bipotential stands for the plastic dissipation power, and thus, is supposed to be positive.

The bipotential concept sheds a new light on known nonassociated laws: Coulomb's dry friction, nonassociated Drucker–Prager law, the modified Clam–Clay model for soil materials and Lemaitre plastic damage model (De Saxcé, 1992; Bodovillé, 1999; Bodovillé and de Saxcé, 2001; Dang Van et al., 2002; Bousshine et al., 2001, 2003). On this ground, an extension of usual bound theorems of the limit analysis was proposed by De Saxcé and Bousshine (1998).

2.2. Variational formulation of shakedown problems with the bipotential

Let be $\boldsymbol{\Omega}$ a solid body with an elastic–plastic material admitting a bipotential:

$$\forall(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\kappa}}') \in V, \quad \forall(\boldsymbol{\sigma}, \boldsymbol{\pi}') \in F, \quad b[(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\kappa}}'), (\boldsymbol{\sigma}, \boldsymbol{\pi}')] \geqslant \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \boldsymbol{\pi}' \cdot \dot{\boldsymbol{\kappa}}' \tag{3}$$

where ":" means the double contracted tensorial product.

It is subjected to variable periodic external actions varying between given limits controlled by a load factor λ. The following question arises: under what conditions does the body shakes down? Now, we claim that time independent residual stress fields exist and characterize the shakedown solution. This crucial assumption has not been up to now rigorously proved for the class of implicit standard materials but we adopt it as working assumption, before verifying its exactness on an example detailed further on. It is all the more reasonable given that a similar assumption was successfully used by Chaaba et al. (1998) for a nonassociated flow rule in soil mechanics admitting a bipotential.

For a given stress response $(\mathbf{x}, t) \mapsto \lambda \boldsymbol{\sigma}^{E_0}(\mathbf{x}, t)$, under prescribed repeated periodic actions, given for any $\mathrm{x} \in \boldsymbol{\Omega}$ and any $t \geqslant 0$, for initial conditions on the residual stress field, $\boldsymbol{\rho}(\mathbf{x}, 0) = \boldsymbol{\rho}_0(\mathbf{x})$, and on the associated variables, $\boldsymbol{\pi}'(\mathbf{x}, 0) = \boldsymbol{\pi}'_0(\mathbf{x})$, the elastoplastic evolution of the structure is well defined by the fields $(\mathbf{x}, t) \mapsto \boldsymbol{\rho}(\mathbf{x}, t)$, $(\mathbf{x}, t) \mapsto \boldsymbol{\pi}'(\mathbf{x}, t)$, $(\mathbf{x}, t) \mapsto \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}(\mathbf{x}, t)$, $(\mathbf{x}, t) \mapsto \dot{\boldsymbol{\kappa}}'(\mathbf{x}, t)$, given for any $\mathbf{x} \in \boldsymbol{\Omega}$ and at any time. After a transient phase, which can be infinite, the fields $\boldsymbol{\rho}$, $\boldsymbol{\pi}'$, $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ and $\dot{\boldsymbol{\kappa}}'$ tend to periodic asymptotic fields. As we want to bypass the transient phase, we are interested only by the asymptotic fields. If shakedown occurs, the plastic strain $\boldsymbol{\varepsilon}^{\mathrm{p}}$ and the other internal variables $\boldsymbol{\kappa}'$ stabilize and the total dissipation is bounded. Then, we assume that the asymptotic solution is characterized by the time independent fields $\mathbf{x}\mapsto\bar{\boldsymbol{\rho}}(\mathbf{x})$ and $\mathbf{x}\mapsto\bar{\boldsymbol{\pi}}'(\mathbf{x})$. On this ground, we define admissible stress fields $(\bar{\boldsymbol{\rho}},\bar{\boldsymbol{\pi}}')$ in the sense that:

(i) $\bar{\boldsymbol{\rho}}$ is a residual stress field;

(ii) $\bar{\boldsymbol{\rho}}$ and $\bar{\boldsymbol{\pi}}'$ are time-independent and plastically admissible when adding to $\bar{\boldsymbol{\rho}}$ the stress response $\boldsymbol{\sigma}^E = \lambda\boldsymbol{\sigma}^{E_0}$ in the corresponding fictitious elastic body:

$$\forall \mathbf{x} \in \boldsymbol{\Omega},\ \forall t, \quad (\boldsymbol{\sigma}^E(\mathbf{x},t) + \bar{\boldsymbol{\rho}}(\mathbf{x}), \bar{\boldsymbol{\pi}}'(\mathbf{x})) = (\lambda\boldsymbol{\sigma}^{E_0}(\mathbf{x},t) + \bar{\boldsymbol{\rho}}(\mathbf{x}), \bar{\boldsymbol{\pi}}'(\mathbf{x})) \in K$$

where K is the elastic domain. It has been showed by Martin (1975), that when the collapse occurs by ratchet or by plastic shakedown, then $\lambda > \lambda^{\mathrm{a}}$. In the other hand, if $\lambda = \lambda^{\mathrm{a}}$, the asymptotic velocity fields $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ and $\dot{\boldsymbol{\kappa}}'$ vanish.[1] To avert this difficulty, we consider the limit of the velocity fields as λ tends to λ^{a} by upper values and under the normalization condition

$$\int_\Omega \oint \boldsymbol{\sigma}^{E_0} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\, \mathrm{d}t\, \mathrm{d}\Omega = 1 \tag{4}$$

Thus, the limit velocity fields do not vanish (Dang Van et al., 2002).We define admissible generalized velocity fields $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}},\dot{\boldsymbol{\kappa}}')$ in the sense that:

(iii) the increment of the plastic strain rate on the load cycle $\boldsymbol{\Delta\varepsilon}^{\mathrm{p}} = \oint \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\,\mathrm{d}t$ is kinematically admissible with zero values of the corresponding displacement increment on the supports.

(iv) $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ is plastically admissible in the following sense:

$$\int_\Omega \oint \boldsymbol{\sigma}^{E} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\, \mathrm{d}t\, \mathrm{d}\Omega > 0$$

The reasons to introduce admissible fields $(\bar{\boldsymbol{\rho}},\bar{\boldsymbol{\pi}}')$ characterized by (i)–(ii), and $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}},\dot{\boldsymbol{\kappa}}')$ characterized by (iii)–(iv) is that a possible variational formulation of shakedown problems arises from introducing the so-called bifunctional:

$$\beta_{\mathrm{S}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}},\dot{\boldsymbol{\kappa}}',\bar{\boldsymbol{\rho}},\bar{\boldsymbol{\pi}}',\lambda) = \int_\Omega \oint \{b[(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}},\dot{\boldsymbol{\kappa}}'),(\bar{\boldsymbol{\rho}} + \lambda\boldsymbol{\sigma}^{E_0},\bar{\boldsymbol{\pi}}')] - \lambda\boldsymbol{\sigma}^{E_0} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} - \bar{\boldsymbol{\pi}}'\cdot\dot{\boldsymbol{\kappa}}'\}\,\mathrm{d}t\,\mathrm{d}\Omega \tag{5}$$

By virtue of the principle of virtual work, one has, for admissible fields:

$$\int_\Omega \oint \bar{\boldsymbol{\rho}} :\ \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\,\mathrm{d}t\,\mathrm{d}\Omega = \int_\Omega \bar{\boldsymbol{\rho}} :\ \boldsymbol{\Delta\varepsilon}^{\mathrm{p}}\,\mathrm{d}\Omega = 0 \tag{6}$$

A straightforward consequence of (3), (5) and (6) is that for any admissible fields:

$$\beta_{\mathrm{S}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}*},\dot{\boldsymbol{\kappa}}'^{*},\bar{\boldsymbol{\rho}}^{*},\bar{\boldsymbol{\pi}}'^{*},\lambda) \geqslant 0 \tag{7}$$

As λ tends to λ^{a} by upper values, the limit velocity field $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}},\dot{\boldsymbol{\kappa}}')$ and the stress field $(\lambda^{\mathrm{a}}\boldsymbol{\sigma}^{E_0} + \bar{\boldsymbol{\rho}},\bar{\boldsymbol{\pi}}')$ satisfy the constitutive law. According to (2), one has:

$$\beta_{\mathrm{S}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}},\dot{\boldsymbol{\kappa}}',\bar{\boldsymbol{\rho}},\bar{\boldsymbol{\pi}}',\lambda^{\mathrm{a}}) = 0 \tag{8}$$

We call $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}},\dot{\boldsymbol{\kappa}}',\bar{\boldsymbol{\rho}},\bar{\boldsymbol{\pi}}',\lambda^{\mathrm{a}})$ the exact solution, in opposition to any other possible admissible one $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}*},\dot{\boldsymbol{\kappa}}'^{*},\bar{\boldsymbol{\rho}}^{*},\bar{\boldsymbol{\pi}}'^{*},\lambda)$. For the exact solution $((\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}},\dot{\boldsymbol{\kappa}}'),(\bar{\boldsymbol{\rho}},\bar{\boldsymbol{\pi}}'))$, condition (8) combined with the normalization condition (4) allows one to calculate the value of the shakedown load factor:

[1] For the particular case of a flow mechanism (as in limit analysis), the collapse occurs for $\lambda = \lambda^{\mathrm{a}}$, but the corresponding velocity fields $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ and $\dot{\boldsymbol{\kappa}}'$ do not vanish, and the previously pointed out difficulty does not exist anymore.

$$\lambda^{\mathrm{a}} = \int_{\Omega} \oint \{ b[(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \dot{\boldsymbol{\kappa}}'), (\lambda^{\mathrm{a}} \boldsymbol{\sigma}^{E_0} + \bar{\boldsymbol{\rho}}, \bar{\boldsymbol{\pi}}')] - \bar{\boldsymbol{\pi}}' \cdot \dot{\boldsymbol{\kappa}}' \} \, \mathrm{d}t \, \mathrm{d}\Omega \tag{9}$$

If the constitutive law is associated, it can be represented by a separate bipotential:

$$b(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}) = \varphi(\dot{\boldsymbol{\kappa}}) + \psi_K(\boldsymbol{\pi})$$

where ψ_K is the indicator function of K, i.e. $\psi_K(\boldsymbol{\pi}) = 0$ if $\pi \in K$ and $\psi_K(\boldsymbol{\pi}) = +\infty$ otherwise, while φ is the support function of K:

$$\varphi(\dot{\boldsymbol{\kappa}}) = \sup_{\pi \in K} (\boldsymbol{\pi} \cdot \dot{\boldsymbol{\kappa}})$$

that is the well-known dissipation function. For the classical case of the elastic perfectly plastic model, the additional internal variables $\boldsymbol{\kappa}'$ have not to be considered. Then, the bifunctional is separated too:

$$\beta_{\mathrm{S}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \bar{\boldsymbol{\rho}}, \lambda) = \Phi_{\mathrm{S}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}) + \Pi_{\mathrm{S}}(\bar{\boldsymbol{\rho}}, \lambda)$$

where

$$\Phi_{\mathrm{S}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}) = \int_{\Omega} \oint \varphi(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}) \, \mathrm{d}t \, \mathrm{d}\Omega$$

and

$$\Pi_{\mathrm{S}}(\bar{\boldsymbol{\rho}}, \lambda) = \int_{\Omega} \oint \psi_K(\lambda \boldsymbol{\sigma}^{E_0} + \bar{\boldsymbol{\rho}}) \, \mathrm{d}t \, \mathrm{d}\Omega - \lambda$$

For any admissible fields, it holds, as consequence of (8) and (9):

$$\Phi_{\mathrm{S}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}*}) \geqslant \Phi_{\mathrm{S}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}) = -\Pi_{\mathrm{S}}(\bar{\boldsymbol{\rho}}, \lambda^{\mathrm{a}}) = \lambda^{\mathrm{a}} \geqslant -\Pi_{\mathrm{S}}(\bar{\boldsymbol{\rho}}^*, \lambda)$$

leading to classical dual optimization problems. The kinematical factor associated to $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}*}$ is defined by Martin (1975):

$$\lambda^{\mathrm{k}} = \Phi_{\mathrm{S}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}*})$$

In short, one has:

$$\lambda^{\mathrm{k}} \geqslant \lambda^{\mathrm{a}} \geqslant \lambda$$

which is the classical expression of the statical and kinematical dual theorems of the shakedown theory.

On the other hand, for the event of a nonassociated constitutive law admitting a bipotential, the bifunctional cannot be split. Thus, the kinematical and statical fields have to be determined together, which, in our opinion, provides a weak form of the classical dual theorems, adapted to the class of implicit standard materials. In other words, the kinematical and statical shakedown problems cannot be solved independently one of each other but they are "coupled".

Remark. If a solution λ^{a} of Eq. (9) is obtained, then $(\lambda^{\mathrm{a}} \boldsymbol{\sigma}^{E_0} + \bar{\boldsymbol{\rho}}, \bar{\boldsymbol{\pi}}') \in K$. As $0 \in K$ and K is convex, for all $0 \leqslant \alpha < 1$:

$$\alpha(\lambda^{\mathrm{a}} \boldsymbol{\sigma}^{E_0} + \bar{\boldsymbol{\rho}}, \bar{\boldsymbol{\pi}}') \in K$$

what means, for all $\lambda = \alpha\lambda^{\mathrm{a}} < \lambda^{\mathrm{a}}$

$$\left(\lambda \boldsymbol{\sigma}^{E_0} + \frac{\lambda}{\lambda^{\mathrm{a}}} \bar{\boldsymbol{\rho}}, \frac{\lambda}{\lambda^{\mathrm{a}}} \bar{\boldsymbol{\pi}}' \right) \in \overset{o}{K}$$

where $\overset{o}{K}$ denotes the interior of K, corresponding to the stress states without plastic yielding.

Thus, for a load factor λ less than the shakedown load λ^{a}, the structure shakes down.

2.3. The plastic flow rule with nonlinear kinematic hardening rule admits a bipotential

For this constitutive law, the additional internal variable velocities $\dot{\boldsymbol{\kappa}}'$ are $(-\dot{\boldsymbol{\alpha}}, -\dot{p})$, where $\dot{\boldsymbol{\alpha}}$ and $\dot{p}$ are respectively the kinematic and isotropic hardening variable rates. The corresponding associated variables $\boldsymbol{\pi}'$ are denoted $(\mathbf{X}, R)$. R and $\mathbf{X}$ are respectively identified to the current threshold and the back-stress.

Let the stress and the elastic domain be defined by:

$$K = \left\{\boldsymbol{\pi} = (\boldsymbol{\sigma}, \mathbf{X}, R) \text{ such that } \sigma_{\text{eq}}(\boldsymbol{\sigma} - \mathbf{X}) - R \leqslant 0\right\} \tag{10}$$

where

$$\sigma_{\text{eq}}(\boldsymbol{\sigma} - \mathbf{X}) = \sqrt{\frac{3}{2}(\boldsymbol{\sigma} - \mathbf{X})'' : (\boldsymbol{\sigma} - \mathbf{X})''}$$

and $(\cdots)''$ means the deviatoric part.

As $R > 0$, the isotropic hardening rule entails that:

$$\varepsilon_{\text{eq}}(\dot{\boldsymbol{\varepsilon}}^{\text{p}}) = \dot{p} \tag{11}$$

where

$$\varepsilon_{\text{eq}}(\dot{\boldsymbol{\varepsilon}}^{\text{p}}) = \sqrt{\frac{2}{3}\dot{\boldsymbol{\varepsilon}}^{\text{p}} : \dot{\boldsymbol{\varepsilon}}^{\text{p}}}$$

The nonassociated kinematic hardening rule introduced by Armstrong and Frederick (1966) and more extensively developed by Lemaitre and Chaboche (1990) and Marquis (1979), can be written as:

$$\dot{\boldsymbol{\alpha}} = \dot{\boldsymbol{\varepsilon}}^{\text{p}} - \frac{3}{2}\frac{\mathbf{X}}{X_\infty}\dot{p} \tag{12}$$

Furthermore, the back stress is linearly dependent on the kinematic variables through:

$$\mathbf{X} = \frac{2}{3}C\boldsymbol{\alpha}$$

where C is a constant kinematic hardening modulus.

It gives a more realistic representation of the cyclic plasticity of metals than Prager's rule and the improved one with a saturation limit surface, in the sense that it better describes the smooth hysteresis shape observed in alternating plastic cycles (Fig. 2). Its main drawback is its nonassociated nature. Nevertheless, as it has been shown in De Saxcé et al. (2000), it admits a bipotential equal to:

$$b(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi}) = \frac{(\sigma_{\text{eq}}(\mathbf{X}))^2}{X_\infty}\dot{p}$$

when (10)–(12) are satisfied and equal to $+\infty$ otherwise.

This function allows us to describe the generalized flow rule through the implicit relation:

$$\dot{\boldsymbol{\kappa}} \in \partial_\pi b(\dot{\boldsymbol{\kappa}}, \boldsymbol{\pi})$$

and the bifunctional (5) then take the form:

$$\beta_{\text{S}}(\dot{\boldsymbol{\varepsilon}}^{\text{p}}, -\dot{\boldsymbol{\alpha}}, -\dot{p}, \bar{\boldsymbol{\rho}}, \mathbf{X}, R, \lambda) = \int_\Omega \oint \left\{\frac{(\sigma_{\text{eq}}(\mathbf{X}))^2}{X_\infty}\dot{p} - \lambda\boldsymbol{\sigma}^{E_0} : \dot{\boldsymbol{\varepsilon}}^{\text{p}} + \mathbf{X} : \dot{\boldsymbol{\alpha}} + R\dot{p}\right\} \mathrm{d}t\, \mathrm{d}\Omega$$

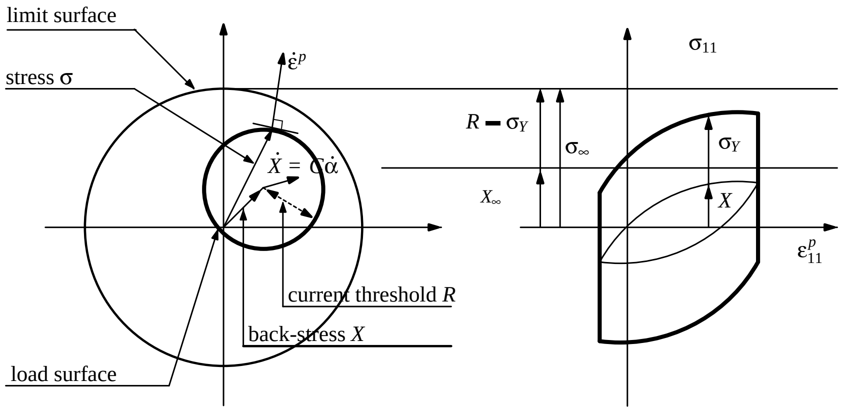


Fig. 2. Nonlinear kinematic hardening rule.

3. Formula for the shakedown factor of a sample under constant tension and alternating cyclic torsion in plane strain state

The analytical example concerns a sample subjected to constant tension σ_{11} and alternating torsion generating a shear stress state σ_{12} in plane strain state.

Only considering the stabilized cycle, the plastic threshold R is supposed to be equal to the constant value σ_Y:

$$R = \sigma_Y$$

Because of the plane strain state, the stress tensor is as follows:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

The back-stress tensor **X** is reduced to its deviatoric part:

$$\mathbf{X} = \begin{bmatrix} X_{11} & X_{12} & 0 \\ X_{12} & -(X_{11}+X_{33}) & 0 \\ 0 & 0 & X_{33} \end{bmatrix}$$

Therefore, the deviatoric part of the shifted stress tensor ($\boldsymbol{\sigma} - \mathbf{X}$) is:

$$(\boldsymbol{\sigma} - \mathbf{X})'' = \begin{bmatrix} \dfrac{2\sigma_{11} - \sigma_{33}}{3} - X_{11} & \sigma_{12} - X_{12} & 0 \\ \sigma_{12} - X_{12} & -\dfrac{\sigma_{11} + \sigma_{33}}{3} + X_{11} + X_{33} & 0 \\ 0 & 0 & \dfrac{2\sigma_{33} - \sigma_{11}}{3} - X_{33} \end{bmatrix}$$

Let us take the following transformed variables:

$$\sigma_{11} = \sigma, X_{11} = X$$
$$\sigma_{12} = \frac{\tau}{\sqrt{3}}, X_{12} = \frac{Y}{\sqrt{3}} \tag{13}$$
$$\sigma_{33} = v\sigma, X_{33} = Z$$

Accounting for Von-Mises criterion, the yield function is of the form:

$$f(\sigma,\tau,X,Y,Z) = (\sigma - X)^2 + (v\sigma - Z)^2 + (\tau - Y)^2 + 2(X+Z)^2 - (\sigma + Z)(v\sigma + X) - \sigma_Y^2$$

such that the yield criterion gives

$$(\sigma - X)^2 + (v\sigma - Z)^2 + (\tau - Y)^2 + 2(X+Z)^2 - (\sigma + Z)(v\sigma + X) \leqslant \sigma_Y^2$$

Assuming incompressibility of plastic strains, the tensor of the plastic strains rates is:

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \begin{bmatrix} \dot{\varepsilon}_{11}^{\mathrm{p}} & \dot{\varepsilon}_{12}^{\mathrm{p}} & 0 \\ \dot{\varepsilon}_{12}^{\mathrm{p}} & -(\dot{\varepsilon}_{11}^{\mathrm{p}} + \dot{\varepsilon}_{33}^{\mathrm{p}}) & 0 \\ 0 & 0 & \dot{\varepsilon}_{33}^{\mathrm{p}} \end{bmatrix}$$

The tensor of the kinematic internal variables rates has the same form:

$$\dot{\boldsymbol{\alpha}} = \begin{bmatrix} \dot{\alpha}_{11} & \dot{\alpha}_{12} & 0 \\ \dot{\alpha}_{12} & -(\dot{\alpha}_{11} + \dot{\alpha}_{33}) & 0 \\ 0 & 0 & \dot{\alpha}_{33} \end{bmatrix}$$

In the same spirit as previously for the stresses, we now take:

$$\dot{\varepsilon}_{11}^{\mathrm{p}} = \dot{\varepsilon}, \quad \dot{\varepsilon}_{12}^{\mathrm{p}} = \frac{\sqrt{3}}{2}\dot{\gamma}, \quad \dot{\varepsilon}_{33}^{\mathrm{p}} = \dot{\eta} \tag{14}$$

then the cumulated plastic deformation becomes:

$$\dot{p} = \varepsilon_{\mathrm{eq}}(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}) = \sqrt{\frac{2}{3}\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} : \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}} = \sqrt{\frac{4}{3}(\dot{\varepsilon}^2 + \dot{\eta}^2 + \dot{\varepsilon}\dot{\eta}) + \dot{\gamma}^2}$$

For the nonlinear kinematic hardening rule (12), we can write:

$$\dot{\alpha}_{11} = \dot{\varepsilon}_{11} - \frac{3}{2}\frac{X_{11}}{X_\infty}\dot{p}$$
$$\dot{\alpha}_{12} = \dot{\varepsilon}_{12} - \frac{3}{2}\frac{X_{12}}{X_\infty}\dot{p}$$
$$\dot{\alpha}_{33} = \dot{\varepsilon}_{33} - \frac{3}{2}\frac{X_{33}}{X_\infty}\dot{p}$$

Putting

$$\dot{\alpha}_{11} = \dot{\alpha}, \quad \dot{\alpha}_{12} = \frac{\sqrt{3}}{2}\dot{\beta}, \quad \dot{\alpha}_{33} = \dot{\xi}$$

and using (13) and (14), one gets the following condensed form:

$$\dot{\alpha} = \dot{\varepsilon} - \frac{3}{2}\frac{X}{X_\infty}\dot{p}$$
$$\dot{\beta} = \dot{\gamma} - \frac{Y}{X_\infty}\dot{p}$$
$$\dot{\xi} = \dot{\eta} - \frac{3}{2}\frac{Z}{X_\infty}\dot{p}$$

For the sets of dual variables $(\boldsymbol{\sigma}, \mathbf{X}, R) \in K$, $(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, -\dot{\boldsymbol{\alpha}}, -\dot{p}) \in K^*$ (the polar cone of K) verifying the nonlinear kinematic hardening rule, the bipotential function reduces to:

$$b[(\boldsymbol{\sigma},\mathbf{X},R),(\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}},-\dot{\boldsymbol{\alpha}},-\dot{p})]=\frac{(\sigma_{\mathrm{eq}}(\mathbf{X}))^2}{X_\infty}\dot{p}=\frac{3X^2+Y^2+3Z^2+3XZ}{X_\infty}\sqrt{\frac{4}{3}(\dot{\varepsilon}^2+\dot{\eta}^2+\dot{\varepsilon}\dot{\eta})+\dot{\gamma}^2} \tag{15}$$

The unit value is taken as reference shear stress. This allows us to identify the load factor as the maximum shear stress. Therefore, considering cyclic loading, the state will alternate between two shear stress extrema such that, for the maximum of the cycle:

$$\begin{aligned} &\sigma=\sigma, \quad \tau=\lambda(\tau^0=1)\\ &\dot{\varepsilon}=\dot{\varepsilon}_+, \quad \dot{\gamma}=\dot{\gamma}_+, \quad \dot{\eta}=\dot{\eta}_+, \quad \dot{\alpha}=\dot{\alpha}_+, \quad \dot{\beta}=\dot{\beta}_+, \quad \dot{\xi}=\dot{\xi}_+ \end{aligned} \tag{16}$$

for the minimum of the cycle, we have:

$$\begin{aligned} &\sigma=\sigma, \quad \tau=-\lambda(\tau^0=-1)\\ &\dot{\varepsilon}=\dot{\varepsilon}_-, \quad \dot{\gamma}=\dot{\gamma}_-, \quad \dot{\eta}=\dot{\eta}_-, \quad \dot{\alpha}=\dot{\alpha}_-, \quad \dot{\beta}=\dot{\beta}_-, \quad \dot{\xi}=\dot{\xi}_- \end{aligned} \tag{17}$$

We would like now to use the general shakedown formulation of Section 2.2. As we consider a sample with uniformly distributed stress and strain fields, the residual stress $\bar{\boldsymbol{\rho}}$ must not be considered. In fact, the back-stresses $\bar{\mathbf{X}}$ play the part of the residual stresses but at a micro-scale smaller than the one of the reference elementary volume. Further, they will be considered as time-independent but we will omit the over bar.

For sake of simplicity, a unit volume sample Ω is now considered, in order to avoid the volume integrals.

3.1. Calculation of the shakedown factor

It is assumed that the collapse occurs by ratchetting only in traction:

$$\oint \dot{\gamma}\,\mathrm{d}t=0$$

We note that nonvanishing contributions to the time integral are only related to the extrema of the collapse cycle. At each one, we consider that the velocities are constant during a unit time interval, which leads to:

$$\dot{\gamma}_+ + \dot{\gamma}_- = 0$$

On the other hand, because of the normalization condition (16) and (17), one has:

$$\oint \tau\dot{\gamma}\,\mathrm{d}t=\lambda\oint \tau^0\dot{\gamma}\,\mathrm{d}t=\lambda(\dot{\gamma}_+-\dot{\gamma}_-)=\lambda \tag{18}$$

Consequently, we have:

$$\begin{aligned} &\dot{\gamma}_+=\dot{\gamma}_-=\frac{1}{2}\\ &\dot{p}=\sqrt{\frac{4}{3}(\dot{\varepsilon}^2+\dot{\eta}^2+\dot{\varepsilon}\dot{\eta})+\frac{1}{4}} \end{aligned} \tag{19}$$

We suppose the maxima of the cycle are located on the load surface:

$$\begin{aligned} &(\sigma-X)^2+(v\sigma-Z)^2+(\lambda-Y)^2+2(X+Z)^2-(\sigma+Z)(v\sigma+X)-\sigma_Y^2=0\\ &(\sigma-X)^2+(v\sigma-Z)^2+(-\lambda-Y)^2+2(X+Z)^2-(\sigma+Z)(v\sigma+X)-\sigma_Y^2=0 \end{aligned}$$

The difference between the two equations gives

$$(\lambda-Y)^2-(-\lambda-Y)^2=-4\lambda Y=0 \tag{20}$$

Because λ is nonnegative, $Y = 0$ and the yield criterion becomes:

$$(\sigma - X)^2 + (v\sigma - Z)^2 + 2(X + Z)^2 - (\sigma + Z)(v\sigma + X) + \lambda^2 = \sigma_Y^2 \tag{21}$$

with the following positive solution

$$\lambda = \sqrt{1 - \frac{1}{\sigma_Y^2}((\sigma - X)^2 + (v\sigma - Z)^2 + 2(X + Z)^2 - (\sigma + Z)(v\sigma + X))} \tag{22}$$

Our goal is now to determine the value of X and Z at collapse accounting for the plastic flow and hardening rules. Therefore, we calculate $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}$ and $\dot{p}$, in order to obtain an explicit expression of X and Z through the hardening rule. The plastic yielding rule gives:

$$\begin{aligned}
\dot{\varepsilon} &= \dot{\zeta}\frac{\partial f}{\partial \sigma} = \dot{\zeta}((2 - v)\sigma - 3X) \\
\dot{\gamma} &= \dot{\zeta}\frac{\partial f}{\partial \tau} = 2\dot{\zeta}\tau \\
\dot{\eta} &= \dot{\zeta}\frac{\partial f}{\partial (v\sigma)} = \dot{\zeta}((2v - 1)\sigma - 3Z)
\end{aligned}$$

In particular, at the extrema of the cycle, we have:

$$\begin{aligned}
\dot{\gamma}_+ &= 2\dot{\zeta}_+\lambda \\
\dot{\gamma}_- &= -2\dot{\zeta}_-\lambda
\end{aligned}$$

Combining with the relation (19), the plastic multiplier is equal to:

$$\dot{\zeta}_\pm = \frac{1}{4\lambda}$$

then taking account of the yield criterion (21) and expression (19) of $\dot{p}$, one has:

$$\begin{aligned}
\dot{\varepsilon}_\pm &= \frac{1}{4\lambda}[(2 - v)\sigma - 3X] \\
\dot{\eta}_\pm &= \frac{1}{4\lambda}[(2v - 1)\sigma - 3Z] \\
\dot{p}_\pm &= \frac{\sigma_Y}{2\lambda}
\end{aligned} \tag{23}$$

On the other hand, the nonlinear kinematic hardening rule allows one to write:

$$\begin{aligned}
\dot{\alpha}_\pm &= \frac{1}{4\lambda}\left[(2 - v)\sigma - 3X\frac{\sigma_\infty}{X_\infty}\right] \\
\dot{\xi}_\pm &= \frac{1}{4\lambda}\left[(2v - 1)\sigma - 3Z\frac{\sigma_\infty}{X_\infty}\right]
\end{aligned} \tag{24}$$

where $\sigma_\infty = X_\infty + \sigma_Y$.

A straightforward consequence of the previous development is:

$$\begin{aligned}
\dot{\alpha}_+ &= \dot{\alpha}_- \\
\dot{\xi}_+ &= \dot{\xi}_-
\end{aligned} \tag{25}$$

As shown by Martin (1975), the actual collapse by ratchetting occurs only for a load factor greater than the shakedown one. After a transient phase, the back-stress field tends to a time periodic solution. In other words, the back-stress increment over the collapse cycle vanishes:

$$\Delta X = \oint \dot{X}\,\mathrm{d}t = C\oint \dot{\alpha}\,\mathrm{d}t = C(\dot{\alpha}_+ + \dot{\alpha}_-) = 0$$
$$\Delta Z = \oint \dot{Z}\,\mathrm{d}t = C\oint \dot{\xi}\,\mathrm{d}t = C(\dot{\xi}_+ + \dot{\xi}_-) = 0$$

Therefore

$$\dot{\alpha}_+ = \dot{\alpha}_- = 0$$
$$\dot{\xi}_+ = \dot{\xi}_- = 0$$

Combining with (25) gives

$$\dot{\alpha}_- = \dot{\alpha}_- = 0$$
$$\dot{\xi}_- = \dot{\xi}_- = 0$$

Consequently, from the expression (24) of $\dot{\alpha}_\pm$ and $\dot{\xi}_\pm$, we deduce the values of the back-stress:

$$X = \frac{(2-v)\sigma X_\infty}{3\sigma_\infty} \qquad Z = \frac{(2v-1)\sigma X_\infty}{3\sigma_\infty} \tag{26}$$

Then, putting them into (22), we find:

$$\lambda = \sigma_Y\sqrt{1 - \frac{\sigma^2}{\sigma_Y^2\sigma_\infty^2}(1-v+v^2)(\sigma_\infty - X_\infty)^2}$$

which leads to the following expression:

$$\lambda = \sigma_Y\sqrt{1 - \frac{\sigma^2}{\sigma_\infty^2}(1-v+v^2)} \tag{27}$$

Remark. The particular case $v = 0$ gives the shakedown factor for a plane stress state. This formula was proposed first by Lemaitre and Chaboche (1990) and latter proved in De Saxcé et al. (2000).

3.2. The previous solution is the exact one

The key idea is to consider the corresponding bifunctional:

$$\beta_\mathrm{S} = \oint \{b[(\boldsymbol{\sigma}, \mathbf{X}, R), (\dot{\boldsymbol{\varepsilon}}^\mathrm{p}, -\dot{\boldsymbol{\alpha}}, -\dot{p})] - \sigma\dot{\varepsilon} - \tau\dot{\gamma} - v\sigma\dot{\eta} + X\dot{\alpha} + Y\dot{\beta} + Z\dot{\xi} + (X+Z)(\dot{\alpha}+\dot{\xi}) + R\dot{p}\}\,\mathrm{d}t$$

and to prove its value is zero. Accounting for expression (15) of the bipotential, and normalization condition (18), we simplify:

$$\beta_\mathrm{S} = \oint \left\{\frac{3X^2 + Y^2 + 3Z^2 + 3XZ}{X_\infty}\dot{p} - \sigma\dot{\varepsilon} - v\sigma\dot{\eta} + X\dot{\alpha} + Y\dot{\beta} + Z\dot{\xi} + (X+Z)(\dot{\alpha}+\dot{\xi}) + R\dot{p}\right\}\mathrm{d}t - \lambda$$

For a stabilized cycle, one has $R = \sigma_Y$, and $Y = 0$ as demonstrated in the previous calculations (20). Then:

$$\beta_S = \oint \left\{ \left[\frac{3X^2 + Y^2 + 3Z^2 + 3XZ}{X_\infty} + \sigma_Y \right] \dot{p} - \sigma\dot{\varepsilon} - v\sigma\dot{\eta} + X\dot{\alpha} + Z\dot{\xi} + (X + Z)(\dot{\alpha} + \dot{\xi}) \right\} dt - \lambda$$

Because the tension stress σ acts as a dead load and the back-stresses X and Z are time independent, one has:

$$\beta_S = \oint \left[\frac{3X^2 + Y^2 + 3Z^2 + 3XZ}{X_\infty} + \sigma_Y \right] \dot{p}\, dt - \sigma \oint \dot{\varepsilon}\, dt - v\sigma \oint \dot{\eta}\, dt + X \oint \dot{\alpha}\, dt + Z \oint \dot{\xi}\, dt + (X + Z) \oint (\dot{\alpha} + \dot{\xi})\, dt - \lambda$$

Taking into account the remark of Section 3.1 concerning the time integrals, one has:

$$\beta_S = \left[\frac{3X^2 + Y^2 + 3Z^2 + 3XZ}{X_\infty} + \sigma_Y \right] (\dot{p}_+ + \dot{p}_-) - \sigma(\dot{\varepsilon}_+ + \dot{\varepsilon}_-) - v\sigma(\dot{\eta}_+ + \dot{\eta}_-) + 2X(\dot{\alpha}_+ + \dot{\alpha}_-) + 2Z(\dot{\xi}_+ + \dot{\xi}_-) + X(\dot{\xi}_+ + \dot{\xi}_-) + Z(\dot{\alpha}_+ + \dot{\alpha}_-) - \lambda$$

With the explicit expressions (23), (24) of $\dot{p}_\pm, \dot{\varepsilon}_\pm, \dot{\eta}_\pm, \dot{\alpha}_\pm, \dot{\xi}_\pm$, previously found, we reduce the bifunctional to:

$$\beta_S = \frac{1}{\lambda} \left\{ -(\sigma - X)^2 - (v\sigma - Z)^2 - 2(X + Z)^2 + (\sigma + Z)(v\sigma + X) + \sigma_Y^2 - \lambda^2 \right\}$$

Finally, as the yield criterion (21) at the extrema of the collapse cycle is satisfied, we prove that the bifunctional vanishes:

$$\beta_S = 0$$

The theoretical considerations show that the previous analytical solution is the exact one.

4. Numerical step-by-step study of a sample under constant tension and alternating torsion

4.1. Stress–strain curve

In this section, the purpose is to establish an incremental strain/stress relationship. The total strain increment $\mathbf{d\varepsilon}$ is assumed to be the sum of the elastic strain increment $\mathbf{d\varepsilon}^e$ and the plastic one $\mathbf{d\varepsilon}^p$:

$$\mathbf{d\varepsilon} = \mathbf{d\varepsilon}^e + \mathbf{d}\varepsilon^p$$

The elastic part satisfies the Hooke's law:

$$\mathbf{d\varepsilon} = \frac{1 + v}{E} \mathbf{d\sigma} - \frac{v}{E} (\mathrm{tr}\, \mathbf{d\sigma}) \mathbf{I}$$

The plastic component is given by Prager's formulation of plastic yielding rule:

- if $f(\boldsymbol{\sigma} - \mathbf{X}) < 0$ then $\mathbf{d\varepsilon}^p = 0$
- else if $f(\boldsymbol{\sigma} - \mathbf{X}) = 0$, $\exists d\zeta \geqslant 0$ such that $\mathbf{d\varepsilon}^p = d\zeta \frac{\partial f}{\partial \boldsymbol{\sigma}}$

The consistency condition $\mathrm{d}f = 0$ provides an expression to compute the plastic multiplier:

$$\mathrm{d}\zeta = \frac{\frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{d}\boldsymbol{\sigma}}{\frac{2}{3} C \frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}} - \frac{C}{X_\infty} \frac{\partial f}{\partial \boldsymbol{\sigma}} : \mathbf{X} \sqrt{\frac{2}{3} \frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}}}}$$

where C and X_∞ are materials constants.

Back-stresses and plastic strains are then computed by numerical integration of the nonlinear kinematic hardening rule:

$$\mathbf{dX} = \frac{2C}{3} \mathbf{d}\boldsymbol{\varepsilon}^{\mathrm{p}} - \frac{C}{X_\infty} \mathbf{X}\, \mathrm{d}p = \frac{2C}{3} \mathrm{d}\zeta \frac{\partial f}{\partial \boldsymbol{\sigma}} - \frac{C}{X_\infty} \mathrm{d}\zeta \mathbf{X} \sqrt{\frac{2}{3} \frac{\partial f}{\partial \boldsymbol{\sigma}} : \frac{\partial f}{\partial \boldsymbol{\sigma}}}$$

and the normality law:

$$\mathbf{d}\boldsymbol{\varepsilon}^{\mathrm{p}} = \mathrm{d}\zeta \frac{\partial f}{\partial \boldsymbol{\sigma}}$$

during the two loading phases: first a tension up to $\sigma_{\max}$ (path OA on Fig. 3) and next, a repeated alternating torsion between $\tau_{\max}$ and $-\tau_{\max}$ (path ABACA on Fig. 3). Stress increments are fixed to 1 MPa for the tension loading and to ± 1 MPa for the alternating torsion one. Back-stress $\mathbf{X}$ is initialized to 0 MPa.

The material selected for numerical implementation is 316L steel, its material constants in cyclic plasticity (Lemaitre and Chaboche, 1990) are: $\sigma_Y = 300$ MPa, $v = 0.3$, $E = 205\,000$ MPa. For a tension intensity of 300 MPa, the analytical shakedown load calculated by (27) is $\lambda = 282.84$ MPa. On Figs. 4 and 5, the shear stress is plotted with respect to ε_{11} and ε_{12}.

As the maximum shear stress $\tau_{\max}$ is less than the shakedown load, we observe that, after a transient elastoplastic regime, the strain–stress curve tends to a linear response that shows the shakedown occurs.

4.2. Numerical detection of shakedown load

We would like to determine numerically the shakedown load for a fixed tension stress. In order to detect this value, we fix a maximum shear stress $\tau_{\max}$ and we perform numerous cycles for this value. When the structure shakes down, the width of the cycle for the shear strain tends to zero, theoretically after an infinite

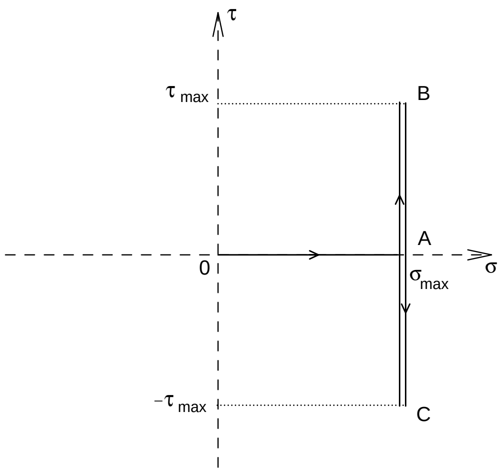


Fig. 3. Loading description.

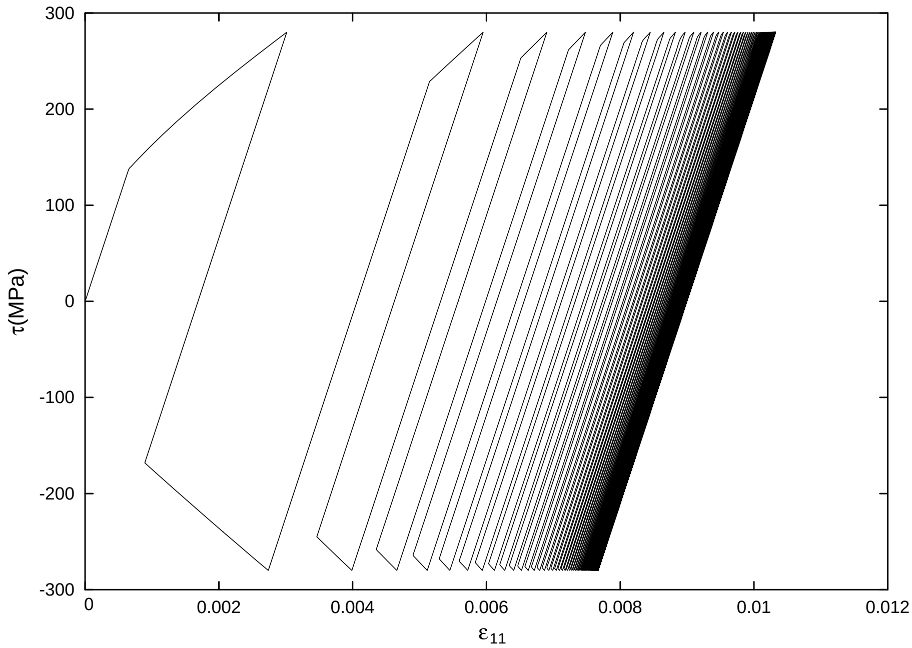


Fig. 4. Axial strain in terms of shear stress for $\sigma = 300$ MPa and $\tau_{max} = 280$ MPa.

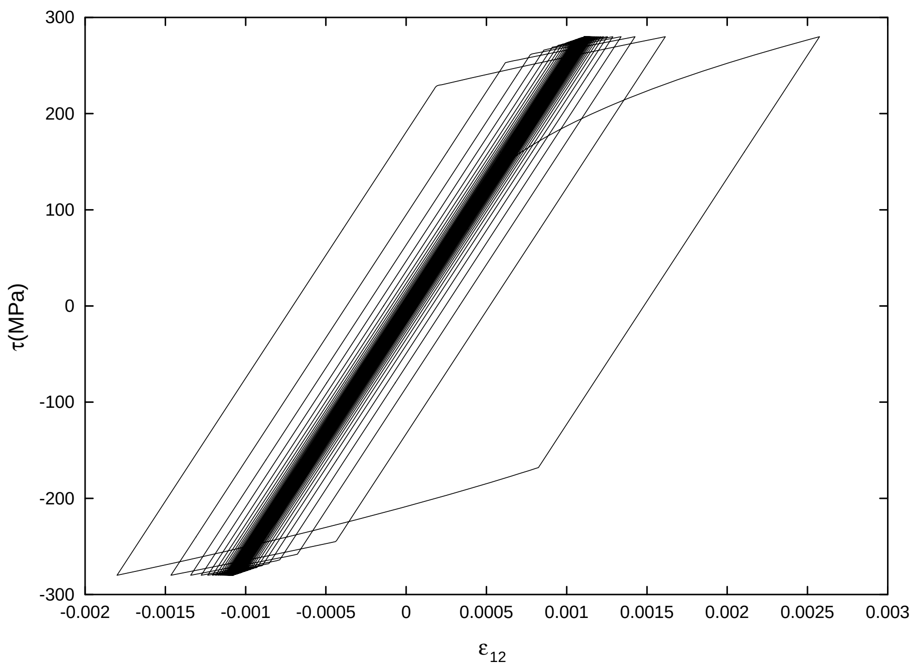


Fig. 5. Shear strain in terms of shear stress for $\sigma = 300$ MPa and $\tau_{max} = 280$ MPa.

number of cycles. In practice, the computation are stopped when the cycle width reaches a given tolerance or when the cycles number become greater than a given maximum. Then, the corresponding maximum shear strain ε^{f}_{max} is considered. This procedure is performed for various values of the maximum shear stress τ_{max}. By plotting the strain values ε^{f}_{max} in terms of maximal shear stress τ_{max} for a tension intensity σ of 100 MPa, we obtain the curve of Fig. 6.

It seems that the sudden sloop modification observed on this curve detects the shakedown load. To confirm this idea, we have computed and plotted the curvature of this curve, by means of the second derivative, in terms of the shear stress (see Fig. 7).

Effectively, we observe a significant pick of the curvature. With this method, the numerical shakedown load for the considered example is 298 MPa, a value closed to the value 298.14 MPa, obtained by analytical formula (27).

By plotting the maximal final shear strain ε^{f}_{max} in terms of maximal shear stress τ_{max} for a tension intensity σ of 250 MPa, we remark on Fig. 8 two picks of minor intensity. The first one corresponds to incipient plastic flow. Between this pick and the second one the shakedown is reached numerically only after two cycles. These two picks do not appear on the preceding curve representing the curvature (Fig. 7) because of the quick transition from elasticity to plasticity and subsequently to shakedown. The same phenomenon is observed for $\sigma = 300$ MPa (see Fig. 9).

For a tension intensity of 350 MPa, only one of the 2 picks of minor intensity remains (see Fig. 10). This can be explained by the fact that the plastification already begins during the tension phase (path OA of Fig. 3).

Remark. Various numerical simulations were performed in order to test the influence of back-stress initialization. Generally speaking, the results have shown a delay of apparition of incipient plastic flow, but no modification of numerical shakedown loads and back-stress values at collapse. As typical example, we consider the case with a tension intensity of 300 MPa and initial back-stresses $X_0 = 10$ MPa, $Y_0 = 2$ MPa

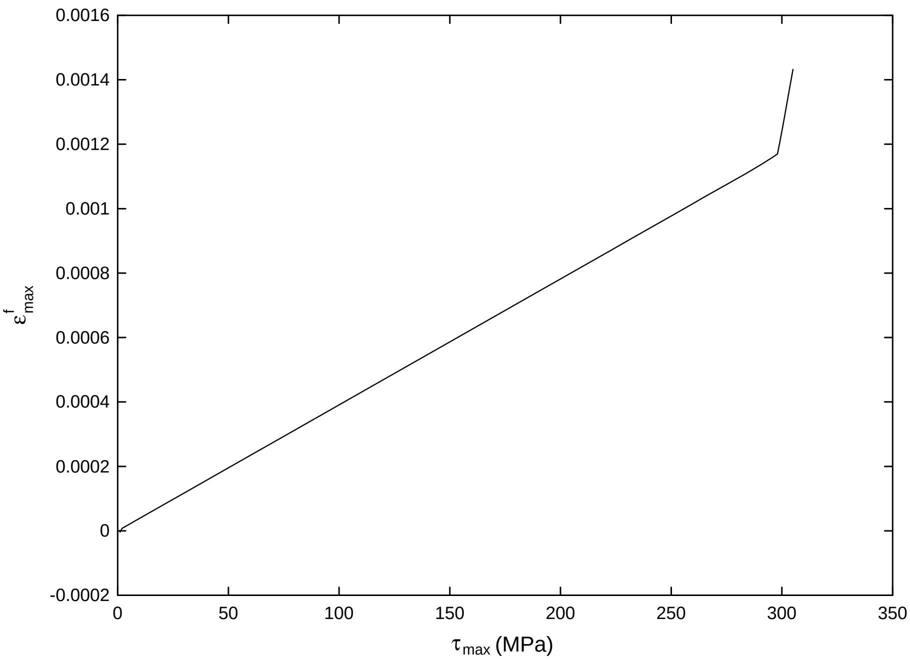


Fig. 6. Maximal final shear strain in terms of shear stress for $\sigma = 100$ MPa.

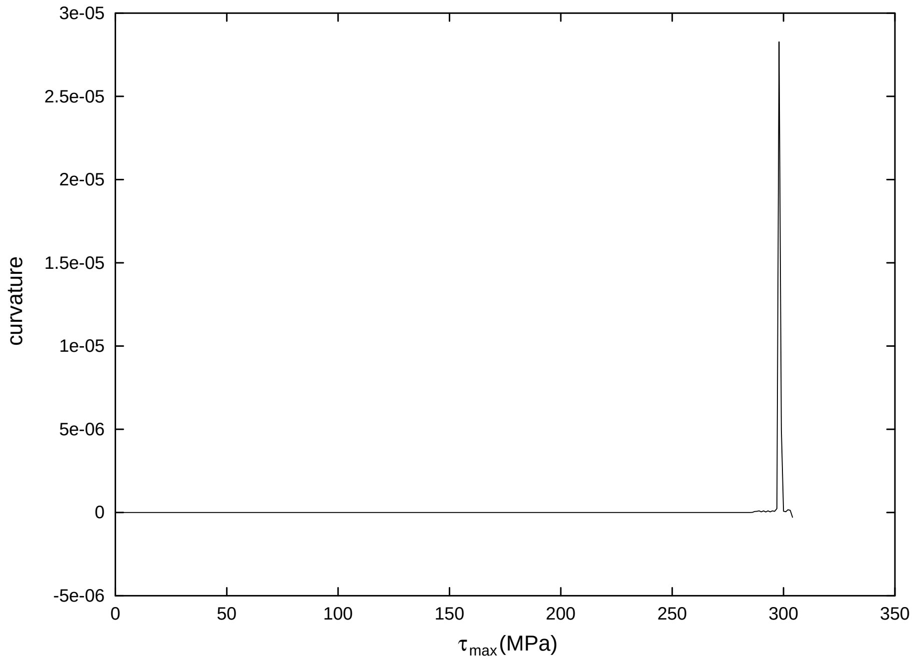


Fig. 7. Curvature of maximal final shear strain–shear stress curve in terms of shear stress for $\sigma = 100$ MPa.

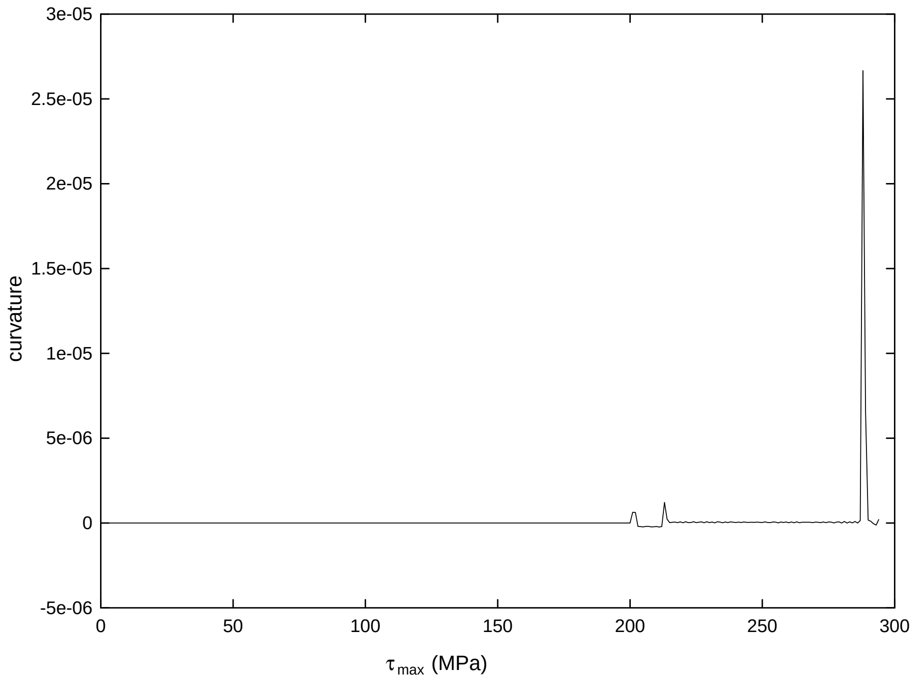


Fig. 8. Curvature of maximal final shear strain–shear stress curve in terms of shear stress for $\sigma = 250$ MPa.

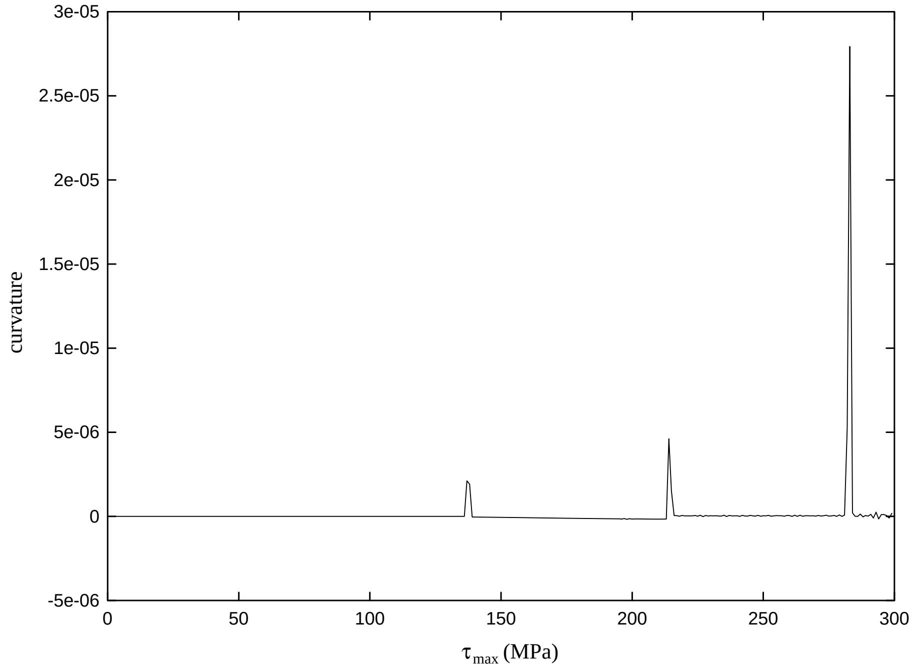


Fig. 9. Curvature of maximal final shear strain–stress curve in terms of shear stress for $\sigma = 300$ MPa.

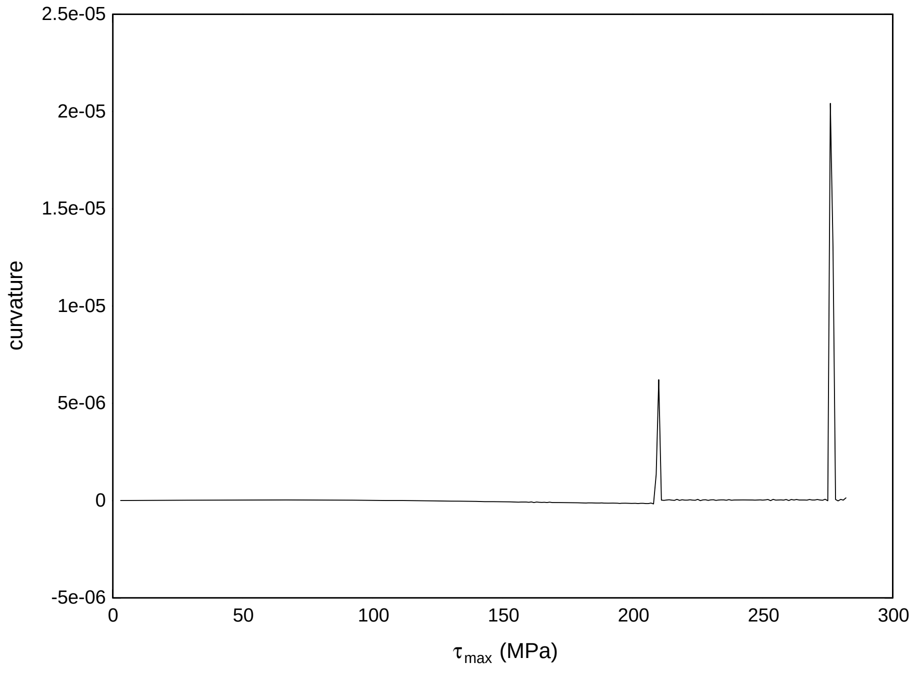


Fig. 10. Curvature of maximal shear strain–shear stress curve in terms of shear stress for $\sigma = 350$ MPa.

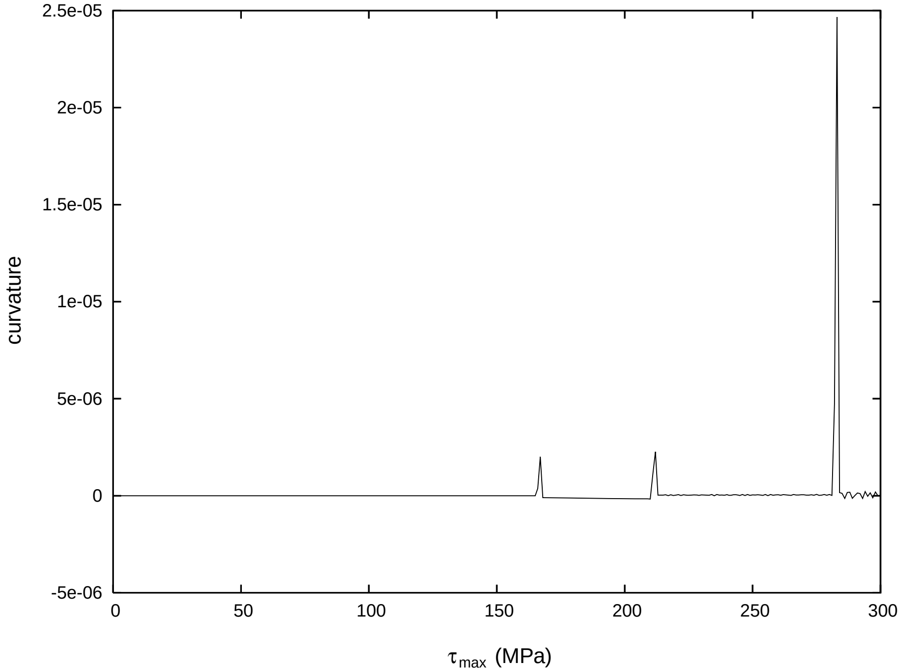


Fig. 11. Curvature of maximal shear strain–shear stress curve in terms of shear stress for $\sigma = 300$ MPa and initial back-stress $X_0 = 10$ MPa, $Y_0 = 2$ MPa and $Z_0 = 1$ MPa.

and $Z_0 = 1$ MPa. We observe that the first pick, corresponding to incipient plastic flow, is detected for a greater value of τ_{max} (compare Figs. 9 and 11), while the values of τ_{max} for the two others picks remain unchanged. Another typical situation occurs when the initial values of back-stress are closed to the asymptotic ones (formula (26)). We obtain a figure similar to Fig. 7, i.e. with only one pick, because of the quick transition from elasticity to plasticity and subsequently to shakedown.

5. Analytical results compared with numerical ones

Applying the numerical method of computing the shakedown load explained in the previous section for various values of the fixed tension, one can compare theoretical shakedown loads calculated by (27) with numerical values (Table 1).

A good agreement can be observed between the theoretical results and the numerical ones, the maximum relative error do not exceeding 0.35%.

Formulae (26) and (20) supply another comparison method: the back-stress values at collapse. Indeed, we can compare the theoretical values to the numerical ones, which are computed for a tension stress fixed and a shear stress equal to the corresponding numerical shakedown previously found. Moreover, we consider that after 1000 cycles the shakedown is reached numerically. As it would be seen in Table 2, there is a good agreement between the theoretical back-stress values at collapse and the numerical ones, confirmed by a maximum relative error of 2.3%.

Using Table 1, we determine the interaction curve limiting the shakedown domain (see Fig. 12). Moreover, for the values of τ_{max} exceeding the shakedown load, we observed that the failure occurs only by ratchet. Alternating plasticity collapse was never obtained.

Table 1
Comparison between analytical and numerical values of shakedown load (for 316L steel) in terms of tension stress value σ (with stress increments fixed to ± 1 MPa for the alternating torsion loading)

σ (MPa)	Analytical shakedown load (MPa)	Numerical shakedown load (MPa)
100	298.143	298
200	292.500	292
300	282.845	283
400	268.747	269
500	249.452	249
600	223.621	224
700	188.584	189
800	137.477	137

Table 2
Comparison between analytical and numerical back-stress values in terms of tension stress value σ (the analytical back-stress Y value is 0.0 MPa (20))

Tension stress	Analytical back stress		Numerical back stress		
σ (MPa)	X (MPa)	Z (MPa)	X (MPa)	Z (MPa)	Y (MPa)
100	35.417	−8.334	34.582	−8.137	0.0063
200	70.834	−16.667	69.455	−16.342	0.0001
300	106.250	−25.000	106.241	−24.997	0.1590
400	141.667	−33.333	141.666	−33.333	0.2533
500	177.083	−41.667	176.652	−41.565	0.0005
600	212.500	−50.000	212.500	−50.000	0.3789
700	247.917	−58.333	247.917	−58.333	0.4155
800	283.333	−66.667	283.177	−66.630	0.0001

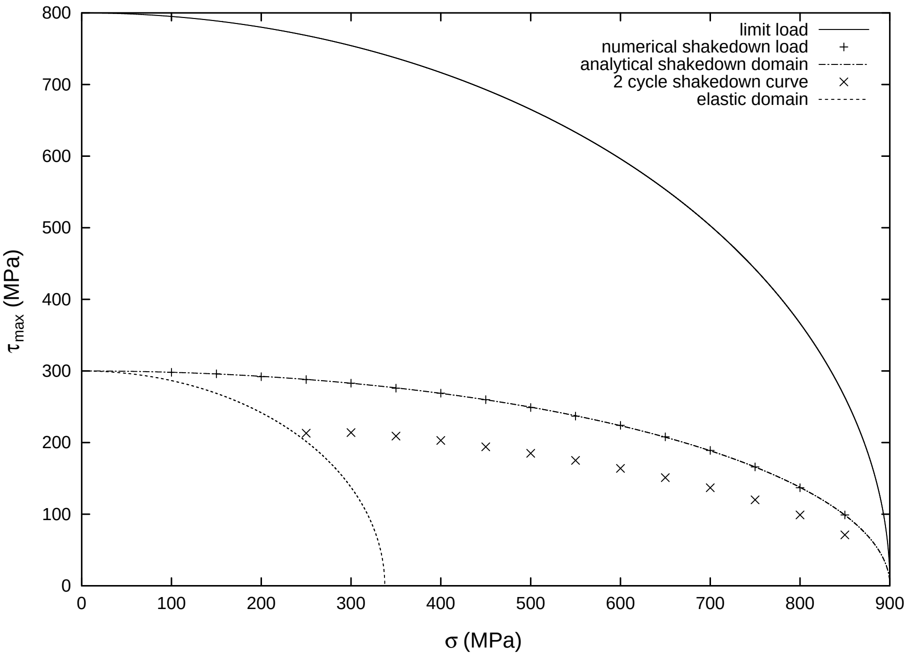


Fig. 12. Interaction curve for 316L steel.

The interaction curve confirms the good agreement between the theoretical and numerical shakedown loads. By adding to this curve the points limiting the field where the shakedown is reached in less than two cycles, a zone that seems to guarantee a better safety than the shakedown field appears.

6. Conclusion

For the problem of a sample under constant tension and alternating cyclic torsion in plane strain conditions, a complete analytical solution was provided and proved to be the exact one according to the bound theorem using bipotential approach (De Saxcé et al., 2000).

For the same problem, a semi-analytical method was implemented to confirm the theoretical results. The good agreement between the analytical solution and the numerical shakedown loads allows us to think that the bipotential approach is of great interest for the study of shakedown problems. With the semi-analytical method, a zone guaranteeing a better safety of the structure than the usual shakedown domain has been highlighted. A theoretical formulation corresponding to this zone should be of practical interest.

In the future, extensions and improvements of the previous results are expected on the followings topics, first a generalization of Melan's theorem to material admitting a bipotential to complete the theoretical foundations, and next, the construction of numerical algorithms based on the bipotential and the mathematical programming to compute the shakedown load.

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